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Quantum switch based on coupled waveguides

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Abstract. The problem of construction of quantum multiplexer is discussed. A possible construction based on resonance transport properties of quantum waveguides coupled through small windows is considered. Small apertures play double role of "connecting channels" and "resonant elements". Transmission coefficients for the system are determined. The workability of the device as a quantum switch to one of three (or to two of three) channels is discussed. Control parameters for the switch are electron energy and bias voltage.

PACS. 72.10.-d Theory of electronic transport; scattering mechanisms – 73.23.Ad Ballistic transport – 02.00 Mathematical methods in physics

1 Introduction

The spectral properties of the Dirichlet Laplacian for coupled strips recently attracted a new wave of interest due to the development of nanoelectronics (see, *e.g.*, [1]). The behaviour of an electron in these novel devices is described by the Schrodinger equation. In many mesoscopic quantum systems the problem reduces to the description of ballistic electron transport, *i.e.* to the Helmholtz equation in a system of waveguides (see, *e.g.*, [2–5]). It has been proved in [6] that the Dirichlet Laplacian for a system of two waveguides Ω_+, Ω_- of widths d_+, d_- coupled laterally through a small window of width $2a$ (Fig. 1) has an eigenvalue k_a^2 close to the threshold and there exist some positive constants c_1, c_2 such that

$$c_1 a^4 \leq \frac{\pi^2}{d_+^2} - \lambda_a \leq c_2 a^4$$

for sufficiently small a (the order of this term was found in [7] on physical level of rigor). Here $d_+ > d_-$. The authors used variational technique and obtained only estimates and not asymptotics. Analogous estimates was obtained for the case of n coupling windows [8]. The asymptotics of the eigenvalue in question was obtained in [9,10]. Method of matching of the asymptotic expansions (in a) for the corresponding solutions was used. The scheme of matching was a modification of that suggested in [11,12]. The difference is that we start from another form of the asymptotic series. One can see that there is some correlation between the result and known weak-coupling asymptotics for Schrodinger operators [13].

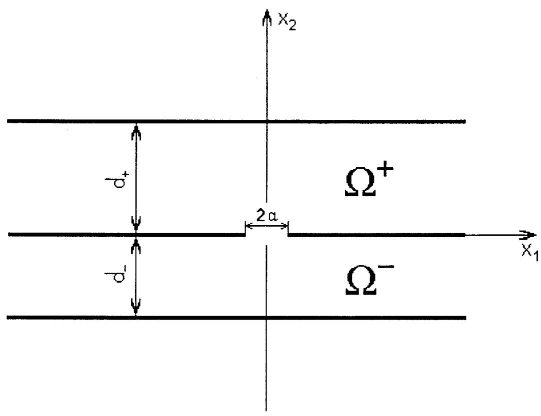

Fig. 1. Two coupled waveguides. Ω_+, Ω_--waveguides of widths d_+, d_- correspondingly, $2a$-width of the aperture.

Analogous asymptotic series is obtained for a resonance (quasibound state) close to Nth threshold. It should be stressed that earlier only the order of the main terms of the asymptotics was obtained by Kunze [7].

A possible construction of three-posed quantum switch based on described resonant property is suggested. Other constructions were described in [14–16].

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2 General mathematical result

We shall obtain the asymptotic expansion of a resonance which tends to the lower bound of the second (third, ..., or other, but not the first) branch of the continuous spectrum (Nth threshold) when $a \to 0$. Let us construct the asymptotic series for the resonance. We shall follow the scheme of matching suggested in [10]. It is related with the approach [12]. But in [12] the author deals with resonances close to the eigenvalue of the unperturbed problem, and we deal with resonances close to the threshold. Due to this fact we have to start from another form of the asymptotic series, and, consequently, modify the whole scheme. The small parameter a be the halfwidth of the opening. Consider the case when $d_+ > d_-$. Let k_a^2 be the resonance close to the point $\frac{\pi^2}{d_-^2}$. We shall seek the asymptotic series of the following form:

$$\left(\frac{\pi^2}{d_-^2} - k_a^2\right)^{1/2} = \sum_{j=2}^{\infty} \sum_{i=0}^{[(j-1)/2]} k_{ji} a^j \left(\log \frac{a}{a_0}\right)^i. \quad (1)$$

For the corresponding eigenfunction $\psi_a(x)$ the asympotic series is the following:

$$\psi_a(x) = \left(\frac{\pi^2}{d_-^2} - k_a^2\right)^{1/2} \sum_{j=0}^{\infty} a^j P_{j+1}\left(D_y, \log\frac{a}{a_0}\right) G^-(x,y,k)|_{y=0},$$
$$x \in \Omega^- \setminus S_{a_0(a/a_0)^{1/2}}, \quad (2)$$

$$\psi_a(x) = \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}(x/a) a^j \left(\log\frac{a}{a_0}\right)^i,$$
$$x \in S_{2a_0(a/a_0)^{1/2}}, \quad (3)$$

$$\psi_a(x) = -\left(\frac{\pi^2}{d_-^2} - k_a^2\right)^{1/2} \sum_{j=0}^{\infty} a^j P_{j+1}\left(D_y, \log\frac{a}{a_0}\right) G^+(x,y,k)|_{y=0},$$
$$x \in \Omega^+ \setminus S_{a_0(a/a_0)^{1/2}}, \quad (4)$$

where a_0 is the natural unit of length, for example, d_-, S_t is a sphere of radius t with the centre at the centre of the opening,

$$v_{ji} \in W^1_{2,\mathrm{loc}}(\Omega^+ \cup \Omega^-),$$

$$P_1\left(D_y, \log\frac{a}{a_0}\right) = a_{10}^{(1)} \frac{\partial}{\partial n_y},$$

P_m are some polynomials in D_y (D_y is a derivative in respect to y):

$$P_m\left(D_y, \log\frac{a}{a_0}\right) = \sum_{q=1}^{m-1} \sum_{i=0}^{[(q-1)/2]} a_{qi}^{(m)} \left(\log\frac{a}{a_0}\right)^i D_y^{m-q+1},$$
$$m \geq 2, \quad (5)$$

$$D_y^{2j+1} = \frac{\partial^{2j+1}}{\partial n_y^{2j+1}}, \quad D_y^{2j} = \frac{\partial^{2j}}{\partial n_y^{2j-1} \partial l_y},$$
$$l = (1,0), n = (0,1),$$

$G^{\pm}$ are the Green functions for the waveguides $\Omega^{\pm}$. It is known that its derivatives can be represented in the close proximity of the point π^2/d_-^2 in the form

$$D_y^j G^{\pm}(x,0,k) = \frac{2}{d_\pm} \sin\frac{\pi x_2}{d_\pm} D_x^j \left(\sin\frac{\pi x_2}{d_\pm}\right)|_{x_2=0} \left(\frac{\pi^2}{d_\pm^2} - k_a^2\right)^{-1/2} + \Phi_j(x,k)\log\frac{r}{a_0} + g_j^{\pm}(x,k)$$
$$+ \sum_{i=0}^{[j/2]} \sum_{t=0}^{j-2i-1} b_{it}^{(j)}(k) r^{-j+2(i+t)} \sin(j-2i)\theta, \quad (6)$$

where (r,θ) are polar coordinates. The terms $b_{it}^{(j)}(k), \Phi_j(x,k), g_j^-(x,k)$ are analytic in respect to k in some neighbourhood of the point π/d_+, $\Phi_j \in C^\infty(\mathbf{R}^2)$ and antisymmetric in respect to x_2, $g_j^{\pm} \in C^\infty(\Omega^{\pm})$,

$$b_{00}^{(j)} = (-1)^{[(j+1)/2]}(j-1)!/\pi, b_{10}^{(3)} = k^2/(2\pi), \Phi_{1n}(0,k) = -k^2/(2\pi). \quad (7)$$

Boundary problems for the terms v_{ji} of the series (3) are obtained by the following way. One substitutes the series (3) and (1) (more precisely, not only (1), but also the corresponding series for k_a) into the Helmholtz equation (for $k = k_a$) with the Dirichlet boundary condition. Then one changes the variables: $\xi = x/a$. The coefficients in the terms with the identical powers of a and $\log a/a_0$ should be equal. Hence, one obtains the following correlations:

$$\Delta_\xi v_{ji} = -\sum_{p=0}^{j-3} \sum_{q=0}^{[p/2]-1} \Lambda_{pq} v_{j-p-2,i-q}, \xi \in \mathbf{R}^2 \setminus \gamma, \quad (8)$$
$$v_{ji} = 0, \xi \in \gamma,$$

where

$$\gamma = \{\xi : \xi_2 = 0, \xi_1 \in (-\infty,-1] \cup [1,\infty)\},$$

and Λ_{pq} are the coefficients of the series

$$k_a^2 = \sum_p \sum_q \Lambda_{pq} a^p \left(\log\frac{a}{a_0}\right)^q.$$

Let $\psi_a^{\pm}(x,k)$ are the series (2), (4), $P_m^{(N)}(D_y, \log\frac{a}{a_0})$ are the sums of type (5) where the summation limit $m-1$ is replaced by $\min(m-1,N)$, $\Psi_{a,N}^{\pm}$ are the series $\psi_a^{\pm}(x,k)$ in which P_j is replaced by $P_j^{(N)}$, $\hat{\psi}_a^{\pm}(x,k), \hat{k}_N(a), \hat{v}_N(\xi,a)$ are the partial sums of the corresponding series. Note that Nth finite sums of the series $\psi_a^{\pm}(x,k)$ and $\Psi_{a,N}^{\pm}(x,k)$ coincide because of the definition of $P_j^{(N)}$. Let us define the operator M_{pq} for the sums $U(x,a)$ of the type (2, 4) (for $k=k_a$) by the following manner: decompose the coefficients of $U(x,a)$ in the asymptotic series for $r\to 0$, replace the variables ($\xi = x/a$) and, simultaneously, replace $\log r$ by $\log\rho + \log a$, $\rho = |\xi|$. Mark as $M_{pq}(U)$ the sum of all terms of the type $a^p(\log\frac{a}{a_0})^q\Phi(\xi)$. Let

$$M_p = \sum_q M_{pq}.$$

The following statement applies place. Let k_a has asymptotics given by (1). Then for $N\ge 1$ the following correlations apply:

$$L_N(\Psi^{\pm}_{a,N}(x,k_a)) = L_{N+1}(\Psi^{\pm}_{a,N}(x,k_a)) + M_{N+1}(\Psi^{\pm}_{a,N+1}(x,k_a))$$

$$L_N(\Psi^{\pm}_{a,N}(x,k_a)) = \sum_{j=1}^{N}\sum_{i=0}^{[(j-1)/2]} V^{\pm}_{ji}(\xi)a^j\left(\log\frac{a}{a_0}\right)^i,$$

$$L_N(\Psi^{\pm}_{a,N}(x,k_a)) - L_N(\widehat{\psi}^{\pm}_{a,N}(x,\hat{k}_N(a))) = O\left(a\rho^{-N} + a^N\left(\log\frac{a}{a_0}\right)^N\rho^{-1}\right),$$

$$(L_N(\Psi^{\pm}_{a,N}(x,k_a)) - L_N(\widehat{\psi}^{\pm}_{a,N}(x,\hat{k}_N(a))))_{\xi_i} = O\left(a\rho^{-N-1} + a^N\left(\log\frac{a}{a_0}\right)^N\rho^{-2}\right),$$

$$\hat{\psi}^{\pm}_{a,N}(x,\hat{k}_a)) - L_N(\hat{\psi}^{\pm}_{a,N}(x,\hat{k}_N(a))) = O\left(r^{N+1} + a^{N+1}\left(\log\frac{a}{a_0}\right)^N\right),$$

$$(\hat{\psi}^{\pm}_{a,N}(x,\hat{k}_a)))_{x_i} - (L_N(\hat{\psi}^{\pm}_{a,N}(x,\hat{k}_N(a)))_{x_i} = O\left(r^N + a^{N+1}\left(\log\frac{a}{a_0}\right)^N/r\right),$$

for $\rho\to\infty, r\to 0$, correspondingly. Series $V^{\pm}_{ji}(\xi)$ does not depend on N, is the asymptotic solution of (8) for $\rho\to\infty$, and $v_{qm} = V^{\pm}_{qm}(\xi)$ in the right part of (8) has a structure:

$$V^{\pm}_{ji}(\xi) = \sum_{q=-p}^{\infty}\rho^{-q}f^{\pm}_{jiq}(\theta) + \log\rho\sum_{q=1}^{p-2}\rho^q F^{\pm}_{jiq}(\theta), \qquad (9)$$

where $p=j-2i$, $F^{\pm}_{jiq}(\theta), f^{\pm}_{jiq}(\theta)$ are linear combinations of $\sin m\theta$,and their sums $V^{-}_{ji}(\xi)+V^{+}_{ji}(\xi)$ are polynomials of $j-2i$ order. Series $V^{\pm}_{Ni}(\xi)$ has the form

$$V^{\pm}_{Ni}(\xi) = \hat{V}^{\pm}_{Ni}(\xi) + k_{N+1,i}k_{20}^{-1}(V^{\pm}_{10}(\xi) \pm \widetilde{V^{\pm}_{10}}(\xi)) \pm \frac{k_{20}}{\pi}\sum_{i=0}^{[(N-1)/2]}\sum_{j=2}^{\infty}a_{Nl}^{N-1+p}(-1)^{[(j+1)/2]}(j-1)!\rho^{-j}\sin j\theta,$$

where $\hat{V}^{\pm}_{Ni}(\xi)$ does not depend on $k_{q+1,p}, a^{(m)}_{qp}$ for $q\ge N$,

$$\widetilde{V^{\pm}_{10}}(\xi) = \begin{cases} 0, \ \xi_2>0,\\ \xi_2, \ \xi_2<0.\end{cases}$$

Note that this statement is analogous to the corresponding lemma in [12] and the proof consists of direct calculations using asymptotics (6, 7). Thus, to make matching it is necessary to show that there exist values k_{ji}, polynomials P_j and functions v_{ji} being solutions of (8) such that the asymptotics of $v_{ji}, \rho\to\infty, \xi_2>0(\xi_2<0)$, coincides with the series $V^{+}_{ji}(\xi)(V^{-}_{ji}(\xi))$, correspondingly. Below we shall confine our attention to the first terms $k_{20}, k_{30}, k_{40}, k_{41}$ only.

The result of matching procedure is as follows. The asymptotics of k_a^2 close to the second threshold is as follows

$$k_a^2 = k_0^2 - k_{20}^2a^4 - 2k_{20}(k_{40}+k_{41}\log\frac{a}{a_0})a^6 - (k_{40}^2 + 2k_{40}k_{41}\log\frac{a}{a_0} + k_{41}^2(\log\frac{a}{a_0})^2)a^8 + o(a^8), \qquad (10)$$

where

$$k_0^2 = \frac{\pi^2}{d_-^2},$$
$$k_{20} = \frac{\pi^3}{4d_-^3}, \quad k_{30}=0,$$
$$k_{40} = \frac{\pi^4}{16d_-^2}(\frac{3\pi}{8d_-^3} - \frac{1}{d_-}(g_x^+ + g_x^-) + \frac{\pi}{d_+^2\sqrt{d_-^2-d_+^2}}),$$
$$g_x^{\pm} = \frac{\partial g_1^{\pm}(x,k)}{\partial x_2}|_{x=0,k=k_0}$$
$$k_{41} = -\frac{\pi^5}{16d_-^5}$$

for the case $d_+>d_-$, and

$$k_0^2 = \frac{\pi^2}{d^2},$$
$$k_{20} = \frac{\pi^3}{2d^3},$$
$$k_{41} = \frac{\pi^4}{4d^3}(-g_x + \frac{3\pi}{16d^2}),$$
$$k_{41} = -\frac{\pi^5}{8d^5}$$

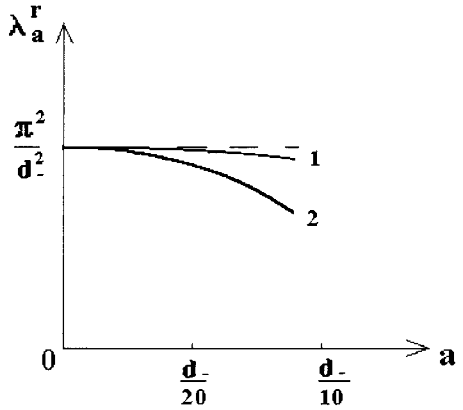

Fig. 2. Position of the real part of the resonance *via* aperture width. 1- for two different waveguides, 2- for two identical waveguides.

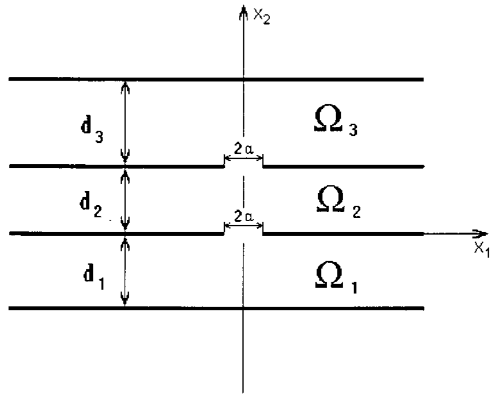

Fig. 3. Geometrical configuration of the system. Ω_1, Ω_2, Ω_3-waveguides.

for the case of two identical waveguides ($d_- = d_+ = d$, $g_x^+ = g_x^- = g_x$).

3 Discussion

The asymptotics of the eigenvalue close to the threshold $\frac{\pi^2}{d_+^2}$ is analogous to (10) with corresponding replacement of the parameters.

It should be stressed that formulas for the case of two identical waveguides are not limits of those for two different waveguides when $d_+ \to d_-$. It is related with a physical reason. One has a different situation below the threshold for these two cases: if $d_- \neq d_+$ then there is only one way for an electron to come out to infinity, if $d_- = d_+$ then there are simultaneously two ways for the electron to come out to infinity. That is why an additional factor 2 appears. The dependence of the real part λ_a^r of the resonance on the width of the aperture is shown in Figure 2.

Consider three parallel coupled waveguides (Fig. 3). Let the central waveguide be narrower than the two others, and the incoming electron in the central waveguide has an energy close to the second thresholds of the two lateral waveguides. One can construct the asymptotics of resonances and the solution of the scattering problem by a way analogous to that for two coupled waveguides. The dependence of the transmission coefficients to each waveguide has resonant character due to the existence of quasi-bound states near the thresholds. It is shown in Figure 4. The width of the resonant peak depends on the diameter $2a$ of the coupling window. The suggested approach gives us the asymptotics in a of the transmission coefficient, *i.e.* we describe the situation for small a. The curves in Figure 4 is for $a = d_-/50$. For greater values of a the peaks are not so sharp. The resonance character of the electron transmission was observed earlier in waveguide systems with attached resonators [17,18]. In our situation the attached waveguide plays the role of the attached resonator.

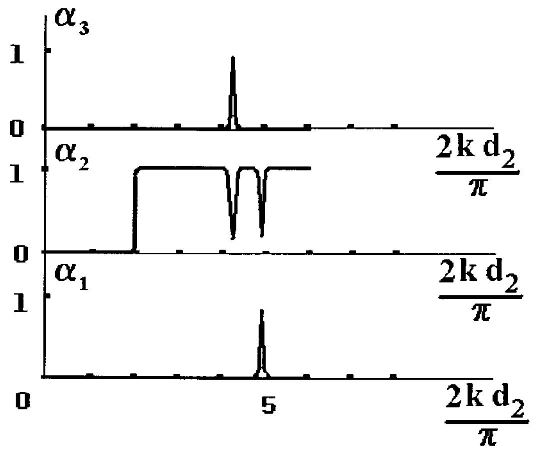

Fig. 4. Dependence of transmission coefficients α_1, α_2, α_3 to waveguide Ω_1, Ω_2, Ω_3, correspondingly, on electron wave number k in dimensionless form.

The effect can be used for the construction of a quantum switch. One can see that when there is a resonant peak in channel 1, the transmission coefficients to other channels are close to zero (and analogously for channel 2). If the electron energy is far from resonances, then one has a transmission coefficient equal to one in Ω_2 and zero transmission in Ω_1, Ω_3. The position of the resonant peaks depends on the width of the corresponding coupling window (see Fig. 2). One can control this parameter by changing bias voltage.

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References

1. C.W.J. Beenakker, H. van Houten, in *Solid State Physics. Advances in Research and Applications*, Vol. 44, edited by H. Ehrenreich, D. Turnbull (Academic Press, San Diego, 1991), pp. 1–228.
2. W. Bulla, F. Gesztesy, W. Renger, B. Simon, Proc. Am. Math. Soc. **125**, 1487 (1997).
3. Y. Takagaki, K. Ploog, Phys. Rev. B **49**, 1782 (1994).
4. P. Duclos, P. Exner, Rev. Math. Phys. **7**, 73 (1995).
5. I.Yu. Popov, S.L. Popova, Europhys. Lett. **24**, 373 (1993).
6. P. Exner, S. Vugalter, Ann. Inst. Henri Poincaré **65**, 109 (1996).
7. Ch. Kunze, Phys. Rev. B **48**, 14338 (1993).
8. P. Exner, S. Vugalter, J. Phys. A **30**, 7863 (1997).
9. I.Yu. Popov, Pisma v Zh. Tech. Fiz. (Tech. Phys. Lett.) **25**, 57 (1999).
10. I.Yu. Popov, Rep. Math. Phys. **43**, 427 (1999).
11. A.M. Ilin, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems* (Nauka, Moscow, 1989).
12. R.R. Gadil'shin, Algebra i Analiz (Leningrad Math. J.) **4**, 88 (1992).
13. B. Simon, Ann. Phys. **97**, 279 (1976).
14. B.S. Pavlov, I.Yu. Popov, V.A. Geyler, O.S. Pershenko, Europhys. Lett. **52**, 196 (2000).
15. B.S. Pavlov, I.Yu. Popov, V.A. Geyler, *ESPRIT 28890 NTCONGS Progress Report (January 1, 1999 — June 30, 1999*) (unpublished).
16. B.S. Pavlov, I.Yu. Popov, O.S. Pershenko, Izv. vuzov, Priborostroenie **43**, 31 (2000).
17. Z. Shao, W. Porod, C.S. Lent, Phys. Rev. B **49**, 7453 (1994).
18. F. Sols, M. Macucci, U. Ravaioli, K. Hess, Appl. Phys. Lett. **54**, 350 (1989).